JOURNAL OF AIRCRAFT
Vol. 43, No. 5, September–October 2006

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Küssner's Function in the Sharp-Edged Gust Problem—A Correction

Altan Kayran*
Middle East Technical University, 06531 Ankara, Turkey

I. Introduction

THE present note aims at formally deriving a formula in the notation of Bisplinghoff[1] suitable for numerically evaluating Küssner's function. As a result of the formal derivation, the reduced formula for the evaluation of Küssner's function given without derivation by Bisplinghoff is corrected. Numerical evaluation of the Küssner function given by Bisplinghoff and derived in the present note has been carried out in the range of 0–20 semichord lengths and compared with the plot of Küssner's function given in the literature, such as in the work of Von Karman and Sears.[2]

Derivation begins by restating the lift per unit span due to a sinusoidal gust as[1]

$$L_{\mathrm{SG}} = 2\pi\rho U b \tilde{\omega}_{\mathrm{SG}}\{H(k) + iJ_1(k)\}e^{iwt} \tag{1a}$$

where

$$H(k) = C(k)[J_0(k) - iJ_1(k)] \tag{1b}$$

and U is the freestream velocity, ρ is the density, $J_0(k)$ and $J_1(k)$ are the Bessel functions of the first kind of order zero and one, k is the reduced frequency, $\tilde{\omega}_{\mathrm{SG}}$ is the amplitude of the sinusoidal gust, w is the frequency, L_{SG} is the lift per unit span due to the sinusoidal gust, and $C(k)$ is the Theodorsen function.[3]

For a thin airfoil, the sharp-edged gust is assumed to strike the airfoil at its leading edge when time t is zero, as shown in Fig. 1.

For a sharp-edged gust moving with a velocity of U the gust vertical velocity is defined as

$$\begin{aligned} \omega_G &= 0, \qquad x > Ut - b \qquad \text{or} \qquad t < (x+b)/U \\ \omega_G &= \omega_0, \qquad x < Ut - b \qquad \text{or} \qquad t > (x+b)/U \end{aligned} \tag{2}$$

Now, express ω_G in terms of the Fourier integral as

$$\omega_G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(w)\exp(iwt)\,\mathrm{d}w \tag{3}$$

$$f(w) = \int_{-\infty}^{\infty} \omega_G(t)\exp(-iwt)\,\mathrm{d}t \tag{4}$$

Instead of taking the Fourier transform of $\omega_G(t)$, take the Fourier transform of

$$F_n = e^{-|t|/n}\omega_G \tag{5}$$

$$F\{F_n\} = \int_{(x+b)/U}^{\infty} \omega_0 \exp\frac{-|t|}{n}\exp(-iwt)\,\mathrm{d}t \tag{6}$$

where the first part of the integral vanishes due to the fact that ω_G is zero in that interval. Taking the integral and substituting in the limits we get

$$F\{F_n\} = \omega_0/[iw + (1/n)]\exp\{-iw[(x+b)/U]\} \times \exp[-|(x+b)/U|/n] \tag{7}$$

At this point, letting $n \to \infty$, the Fourier transform of the sharp-edged gust function can be determined as

$$f(w) = (\omega_0/iw)\exp\{-iw[(x+b)/U]\} \tag{8}$$

The variable $\omega_G(t)$ can now be expressed by the inverse Fourier transform as

$$\omega_G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\omega_0}{iw}\exp\left(-iw\frac{x+b}{U}\right)\exp(iwt)\,\mathrm{d}w \tag{9}$$

where the path of integration must make take an infinitesimal loop below the origin. Defining the reduced frequency k, distance traveled in semichords s, and nondimensional coordinate x^* as

$$k = wb/U \qquad s = Ut/b \qquad x^* = x/b$$

equation (9) can be brought into the following form:

$$\omega_G = \frac{\omega_0}{2\pi}\int_{-\infty}^{\infty}\frac{\exp[ik(s - x^* - 1)]}{ik}\,\mathrm{d}k \tag{10}$$

Notice that in the case of a sinusoidal gust moving at a velocity of U, the vertical velocity ω_{SG} can be expressed as a function of coordinate x and time t as

$$\omega_{\mathrm{SG}}(x - Ut) = \tilde{\omega}_{\mathrm{SG}}\exp\{iw[t - (x/U)]\} = \tilde{\omega}_{\mathrm{SG}}\exp[ik(s - x^*)] \tag{11}$$

and the lift due to sinusoidal gust is given by Eq. (1a). For the sharp-edged gust, comparing Eq. (11) and Eq. (10) and utilizing the superposition integral for linear systems, we can express the lift as

$$L = \frac{\omega_0}{2\pi}\int_{-\infty}^{\infty} L_{\mathrm{SG}}\frac{e^{-ik}}{ik}\,\mathrm{d}k \tag{12}$$

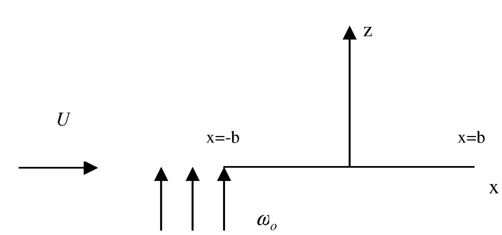

Fig. 1 Thin airfoil encountering a sharp-edged gust.

Received 28 September 2005; revision received 28 November 2005; accepted for publication 29 November 2005.

*Associate Professor, Department of Aerospace Engineering; akayran@metu.edu.tr. Associate Member AIAA.

Bisplinghoff[1] expresses the term inside the braces in Eq. (1a) as

$$F_{SG}(k) + iG_{SG} = C(k)[J_0(k) - iJ_1(k)] + iJ_1(k) \tag{13}$$

With this definition the lift due to a sharp-edged gust given by Eq. (12) can be brought into the form

$$L = 2\pi\rho U b\omega_0 \psi(s) \tag{14}$$

where

$$\psi(s) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{[F_{SG}(k) + iG_{SG}(k)]}{k} \exp[ik(s-1)]\,\mathrm{d}k \tag{15}$$

is the Küssner function. Bisplinghoff reduces the integral in Eq. (15) into a form suitable for numerical evaluation as[1]

$$\psi(s) = \frac{2}{\pi}\int_{0}^{\infty} \frac{[F_{SG}(k) - G_{SG}(k)]\sin ks \sin k}{k}\,\mathrm{d}k \tag{16}$$

for $s > 0$.

In this note it is shown that this equation actually does not represent the true Küssner function and an integral formula suitable for evaluating the Küssner function is derived. Before the derivation it is noted that Küssner's function is zero for $s \leq 0$ because the sharp-edged gust has not come into contact with the airfoil, and we expect the imaginary part of the function to vanish for all values of s.

II. Derivation of Küssner Function Suitable for Numerical Evaluation

Expand $e^{ik(s-1)}$ in terms of sine and cosine functions and express Eq. (15) in terms of real and imaginary parts as

$$\psi(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{R}{k} - i\frac{I}{k}\right)\mathrm{d}k \tag{17}$$

where

$$R = F_{SG}(k)\sin k(s-1) + G_{SG}(k)\cos k(s-1) \tag{18}$$

$$I = F_{SG}(k)\cos k(s-1) - G_{SG}(k)\sin k(s-1) \tag{19}$$

Theodorsen's function $C(k)$ is known as the complex admittance function for steady state oscillations of a linear system. Per Bisplinghoff,[1] any such function is known to have an even real part $F(k)$ and an odd imaginary part $G(k)$. Considering the evenness and oddness of the real and imaginary part of the Theodorsen's function, from Eq. (13) we can show that $F_{SG}(k)$ is an even and $G_{SG}(k)$ is an odd function of the reduced frequency. In that case, both integrands of the imaginary part of Eq. (17) turn out to be odd, yielding

$$\int_{-\infty}^{\infty}\frac{I}{k}\,\mathrm{d}k = 0 \tag{20}$$

Thus, Küssner's function, Eq. (17) becomes

$$\psi(s) = \psi_1(s) + \psi_2(s) \tag{21a}$$

where

$$\psi_1(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F_{SG}(k)\sin k(s-1)}{k}\,\mathrm{d}k \tag{21b}$$

$$\psi_2(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{G_{SG}(k)\cos k(s-1)}{k}\,\mathrm{d}k \tag{21c}$$

Because $F_{SG}(k)$ is even and $G_{SG}(k)$ is odd, the integrands in Eqs. (21a) and (21b) are both even. Thus, it is possible to write the Küssner function as

$$\psi(s) = \psi_{11}(s) + \psi_{22}(s) \tag{22a}$$

where

$$\psi_{11}(s) = \frac{1}{\pi}\int_{0}^{\infty}\frac{F_{SG}(k)\sin k(s-1)}{k}\,\mathrm{d}k \tag{22b}$$

$$\psi_{22}(s) = \frac{1}{\pi}\int_{0}^{\infty}\frac{G_{SG}(k)\cos k(s-1)}{k}\,\mathrm{d}k \tag{22c}$$

Until now, nothing has been stipulated on the Küssner function for $s \leq 0$. By requiring that $\psi(s)$ be zero for $s \leq 0$, Eq. (22a) can further be simplified. By expanding $\sin k(s-1)$ and $\cos k(s-1)$, Küssner's function can be expressed as

$$\psi(s) = \psi_{11s} - \psi_{11c} + \psi_{22s} + \psi_{22c} \tag{23a}$$

where

$$\psi_{11s}(s) = \frac{1}{\pi}\int_{0}^{\infty}F_{SG}(k)\frac{\sin ks\cos k}{k}\,\mathrm{d}k \tag{23b}$$

$$\psi_{11c}(s) = \frac{1}{\pi}\int_{0}^{\infty}F_{SG}(k)\frac{\cos ks\sin k}{k}\,\mathrm{d}k \tag{23c}$$

$$\psi_{22s}(s) = \frac{1}{\pi}\int_{0}^{\infty}G_{SG}(k)\frac{\sin ks\sin k}{k}\,\mathrm{d}k \tag{23d}$$

$$\psi_{22c}(s) = \frac{1}{\pi}\int_{0}^{\infty}G_{SG}(k)\frac{\cos ks\cos k}{k}\,\mathrm{d}k \tag{23e}$$

For $s \leq 0$,

$$\sin ks = -\sin k|s|$$

$$\cos ks = \cos k|s|$$

Now impose $\psi(s) = 0$:

$$\begin{aligned} 0 = &-\frac{1}{\pi}\int_{0}^{\infty}F_{SG}(k)\frac{\sin k|s|\cos k}{k}\,\mathrm{d}k \\ &-\frac{1}{\pi}\int_{0}^{\infty}F_{SG}(k)\frac{\cos k|s|\sin k}{k}\,\mathrm{d}k \\ &+\frac{1}{\pi}\int_{0}^{\infty}G_{SG}(k)\frac{\cos k|s|\cos k}{k}\,\mathrm{d}k \\ &+\frac{1}{\pi}\int_{0}^{\infty}G_{SG}(k)\frac{\sin k|s|\sin k}{k}\,\mathrm{d}k \end{aligned} \tag{24}$$

However, Eq. (24) also holds true for $s > 0$. Thus, we can bring Eq. (24) into a compact form by utilizing the trigonometric identities for $\sin(A+B)$ and $\cos(A+B)$. This yields

$$\begin{aligned} &\frac{1}{\pi}\int_{0}^{\infty}\frac{F_{SG}(k)}{k}\sin k(s+1)\,\mathrm{d}k \\ &-\frac{1}{\pi}\int_{0}^{\infty}\frac{G_{SG}(k)}{k}\cos k(s+1)\,\mathrm{d}k = 0 \end{aligned} \tag{25}$$

Further simplification is possible if Eq. (22a) and Eq. (25) are added together. This operation yields, after simplification,

$$\psi(s) = \frac{2}{\pi}\int_{0}^{\infty}\frac{FG(k)}{k}\sin ks\,\mathrm{d}k \tag{26a}$$

where

$$FG(k) = F_{SG}(k)\cos k + G_{SG}(k)\sin k \tag{26b}$$

Comparing Eq. (16) and Eq. (26a), one can see that they are not identical. The functions defined by $F_{SG}(k)$ and $G_{SG}(k)$ are the real

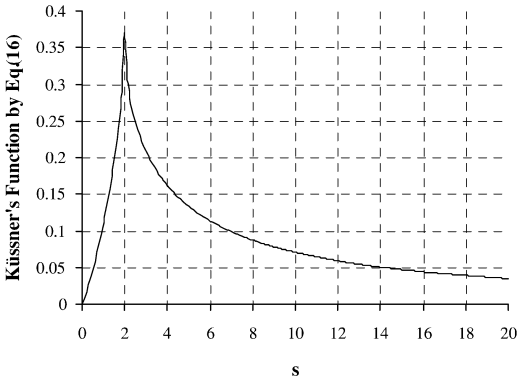

Fig. 2 Küssner's function by Eq. (16) of Bisplinghoff.[1]

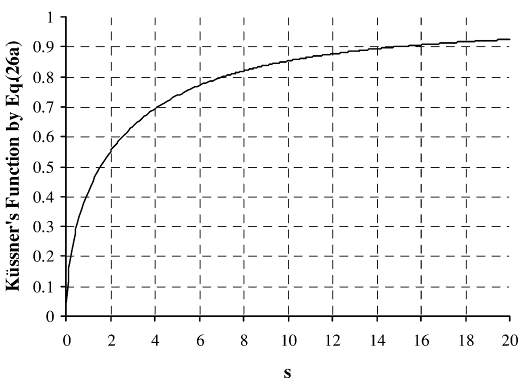

Fig. 3 Küssner's function by Eq. (26a) of the present derivation.

and imaginary parts of Eq. (13) and they tend to zero as reduced frequency goes to infinity. Thus, the integrals given by Eq. (16) and Eq. (26a) converge. Numerical evaluation of the integrals given by Eq. (16) and Eq. (26a) can be carried out using any standard numerical integration scheme. Numerical integration of both integrals has been carried out by the trapezoidal rule. Figure 2 gives the variation of $\psi(s)$ using Eq. (16), and Fig. 3 gives the variation of $\psi(s)$ using Eq. (26a) up to 20 semichords of distance traveled after the first encounter of the thin airfoil with the sharp-edged gust. During the numerical integration, in accordance with the definition of the Fourier transform given by Eq. (9), a small gap is left at the origin that corresponds to zero reduced frequency.

It is noted that Fig. 3 is the actual true Küssner function reported in the literature, such as the one given by Von Karman and Sears.[2] Equation (13), given by Bisplinghoff, which corresponds to Eq. (5-381b) in his book,[1] is asserted to be incorrect. Whether the incorrect equation (16) given in Ref. 1 is a typographical error or is indeed an incorrect mathematical derivation is not clear. Therefore, the reason for this error is unknown.

In addition, Bisplinghoff also gives the plot of Küssner's function overlaid on the plot of Wagner's functionon, Fig. 5-21 of his book.[1] This figure also shows the true Küssner function shown in Fig. 3 of the present note, verifying the erroneous functional relationship for $\psi(s)$ given by Eq. (13).

III. Conclusions

The present note aimed at formally deriving an alternative formula for the determination of Küssner's function suitable for numerical integration, which is used to calculate the lift on a thin airfoil following the entrance of the airfoil into a sharp-edged gust, following the similar notation used by Bisplinghoff. Equation (26a) is claimed to be the true representation of the Küssner function, contrary to Eq. (13), which is given by Bisplinghoff.[1] Because the book on aeroelasticity by Bisplinghoff is a widely referenced textbook in the field, this note is prepared to correct the expression for the Küssner function in Eq. (13) of Bisplinghoff. The correct expression should be the one given by Eq. (26a), which can further be used to determine the lift on airfoils due to arbitrary vertical gusts.

References

[1]Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., "Arbitrary Motion of Thin Airfoils in Incompressible Flow: The Gust Problem," *Aeroelasticity*, Dover, New York, 1996, pp. 285–288.

[2]Von Karman, T., and Sears, W. R., "Airfoil Theory for Non-Uniform Motion," *Journal of the Aeronautical Sciences*, Vol. 5, No. 10, 1938, pp. 379–390; reprinted in Faeth, G. M. (ed.), *Centennial of Powered Flight: A Retrospective of Aerospace Research*, AIAA, Reston, VA, 2003, pp. 5–16.

[3]Theodorsen, T., "General Theory of Aerodynamic Instability and the Mechanism of Flutter," NACA Rept. 496, 1938.